Exercise 63

(a) If $f(x) = x\sqrt{5-x}$, find $f'(x)$.

(b) Find equations of the tangent lines to the curve $y = x\sqrt{5-x}$ at the points $(1, 2)$ and $(4, 4)$.

(c) Illustrate part (b) by graphing the curve and tangent lines on the same screen.

(d) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

Solution

Part (a)

Calculate the derivative of $f(x)$ by using the chain and product rules.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(x\sqrt{5-x}\right) \\
&= \left[\frac{d}{dx}(x)\right]\sqrt{5-x} + x\left[\frac{d}{dx}\left(\sqrt{5-x}\right)\right] \\
&= (1)\sqrt{5-x} + x\left[\frac{1}{2}(5-x)^{-1/2}\cdot\frac{d}{dx}(5-x)\right] \\
&= \sqrt{5-x} + x\left[\frac{1}{2}(5-x)^{-1/2}\cdot(-1)\right] \\
&= \sqrt{5-x} - \frac{x}{2\sqrt{5-x}} \\
&= \frac{2(5-x)-x}{2\sqrt{5-x}} \\
&= \frac{10-3x}{2\sqrt{5-x}}
\end{aligned}$$

Part (b)

Find the slopes of the tangent lines at $(1, 2)$ and $(4, 4)$ by plugging in $x = 1$ and $x = 4$.

$$f'(1) = \frac{10-3(1)}{2\sqrt{5-1}} = \frac{7}{4}$$

$$f'(4) = \frac{10-3(4)}{2\sqrt{5-4}} = \frac{-2}{2} = -1$$

Use the point-slope formula to obtain the equations of these tangent lines.

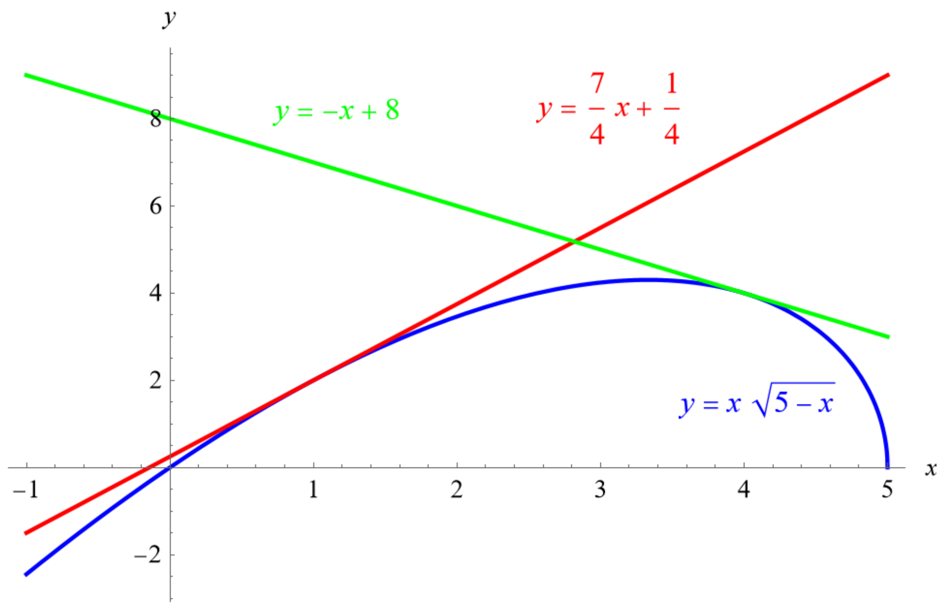
$$y - 2 = \frac{7}{4}(x - 1) \qquad\qquad y - 4 = -1(x - 4)$$

$$y - 2 = \frac{7}{4}x - \frac{7}{4} \qquad\qquad y - 4 = -x + 4$$

$$y = \frac{7}{4}x + \frac{1}{4} \qquad\qquad y = -x + 8$$

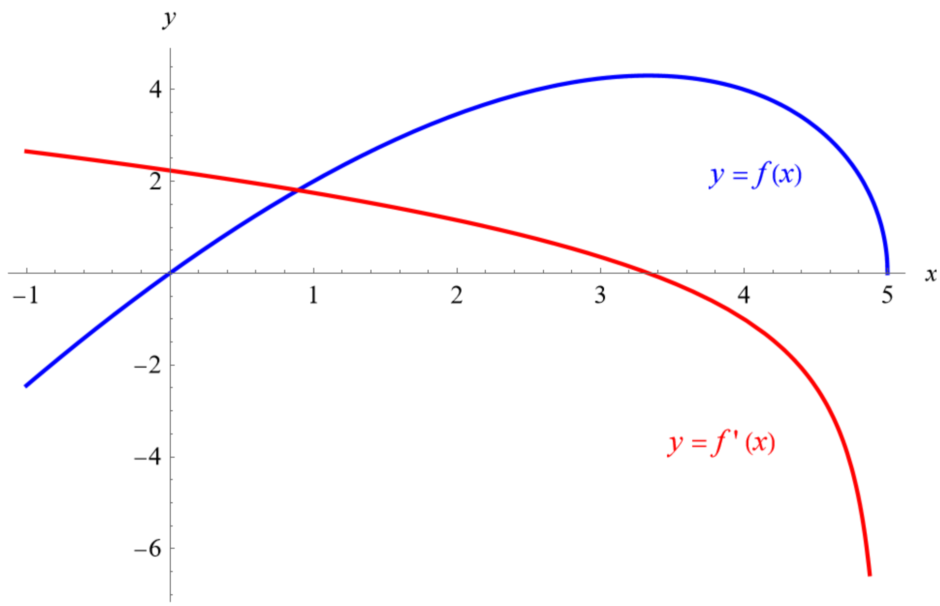
Part (c)

Below is a graph of the curve and its tangent lines at $(1, 2)$ and $(4, 4)$.



Part (d)

Below is a graph of the curve and its derivative.



Notice that $f'(x)$ is zero whenever the tangent line to $f(x)$ is horizontal, $f'(x)$ is positive whenever $f(x)$ is increasing, and $f'(x)$ is negative whenever $f(x)$ is decreasing.